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Full proof of the existence of a degree 8 circulant graph of order $L(8, k)$ of arbitrary diameter k

R. R. Lewis
Department of Mathematics and Statistics
The Open University
Milton Keynes, UK
robert.lewis@open.ac.uk

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This is the full proof of Theorem 3 in the paper “The degree-diameter problem for circulant graphs of degree 8 and 9” by the author [2]. To avoid the paper being unduly long it includes only the exceptions for the orthant of \mathbf{v}_1 for diameter $k \equiv 0 \pmod{2}$ and for $k \equiv 1 \pmod{2}$. In the version below the exceptions for all eight orthants for diameter $k \equiv 0$ and $k \equiv 1 \pmod{2}$ are included in full. This proof closely follows the approach taken by Dougherty and Faber in their proof of the existence of the degree 6 graph of order $DF(6, k)$ for all diameters $k \geq 2$ [1].

Theorem 3. *For all $k \geq 2$, there is an undirected Cayley graph on four generators of a cyclic group which has diameter k and order $L(8, k)$, where*

$$L(8, k) = \begin{cases} (k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{if } k \equiv 0 \pmod{2} \\ (k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{if } k \equiv 1 \pmod{2} \end{cases}$$

Moreover for $k \equiv 0 \pmod{2}$ a generator set is $\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k)/4\}$,
and for $k \equiv 1 \pmod{2}$, $\{1, (k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

Proof. We will show the existence of four-dimensional lattices $L_k \subseteq \mathbb{Z}^4$ such that \mathbb{Z}^4/L_k is cyclic, $S_k + L_k = \mathbb{Z}^4$, where S_k is the set of points in \mathbb{Z}^4 at a distance of at most k from the origin under the l^1 (Manhattan) metric, and $|\mathbb{Z}^4 : L_k| = L(8, k)$ as specified in the theorem. Then, by Theorem 1 of [2], the resultant Cayley graph has diameter at most k .

$$\text{Let } a = \begin{cases} k/2 & \text{for } k \equiv 0 \pmod{2} \\ (k+1)/2 & \text{for } k \equiv 1 \pmod{2}. \end{cases}$$

For $k \equiv 0 \pmod{2}$ let L_k be defined by four generating vectors as follows:

$$\begin{aligned} \mathbf{v}_1 &= (-a - 1, a + 1, a, -a + 1) \\ \mathbf{v}_2 &= (a - 1, a + 1, a + 1, -a) \\ \mathbf{v}_3 &= (-a - 1, -a + 1, a + 1, -a) \\ \mathbf{v}_4 &= (-a, -a, a, a + 1) \end{aligned}$$

Then the following vectors are in L_k :

$$\begin{aligned} -(2a^2 + 2a + 1)\mathbf{v}_1 + (2a^2 + a + 2)\mathbf{v}_2 - (a + 2)\mathbf{v}_3 + \mathbf{v}_4 &= (4a^3 + 4a^2 + 6a + 1, -1, 0, 0), \\ -(2a^3 - 1)\mathbf{v}_1 + (2a^3 - a^2 + 2a - 2)\mathbf{v}_2 - (a^2 + a - 1)\mathbf{v}_3 + (a - 1)\mathbf{v}_4 &= (4a^4 + 4a^2 - 4a, 0, -1, 0), \\ -2a^3\mathbf{v}_1 + (2a^3 - a^2 + 2a - 1)\mathbf{v}_2 - (a^2 + a - 1)\mathbf{v}_3 + (a - 1)\mathbf{v}_4 &= (4a^4 + 4a^2 - 2a, 0, 0, -1) \end{aligned}$$

Hence we have $\mathbf{e}_2 = (4a^3 + 4a^2 + 6a + 1)\mathbf{e}_1$, $\mathbf{e}_3 = (4a^4 + 4a^2 - 4a)\mathbf{e}_1$ and $\mathbf{e}_4 = (4a^4 + 4a^2 - 2a)\mathbf{e}_1$ in \mathbb{Z}^4/L_k , and so \mathbf{e}_1 generates \mathbb{Z}^4/L_k .

$$\text{Also } \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 + 8a^3 + 12a^2 + 4a & 0 & 0 & 0 \\ 4a^3 + 4a^2 + 6a + 1 & -1 & 0 & 0 \\ 4a^4 + 4a^2 - 4a & 0 & -1 & 0 \\ 4a^4 + 4a^2 - 2a & 0 & 0 & -1 \end{pmatrix}$$

$= -(8a^4 + 8a^3 + 12a^2 + 4a) = -(k^4 + 2k^3 + 6k^2 + 4k)/2 = -L(8, k)$, as in the statement of the theorem.

Thus \mathbb{Z}^4/L_k is isomorphic to $\mathbb{Z}_{L(8,k)}$ via an isomorphism taking $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ to $1, 4a^3 + 4a^2 + 6a + 1, 4a^4 + 4a^2 - 4a, 4a^4 + 4a^2 - 2a$. As $a = k/2$ this gives the first generator set specified in the theorem: $\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k)/4\}$.

Similarly for $k \equiv 1 \pmod{2}$ let L_k be defined by four generating vectors as follows:

$$\begin{aligned} \mathbf{v}_1 &= (-a + 1, a + 1, -a + 1, a) \\ \mathbf{v}_2 &= (a + 1, a + 1, -a + 2, a - 1) \\ \mathbf{v}_3 &= (-a - 1, a - 1, a - 1, -a) \\ \mathbf{v}_4 &= (-a, a, a, a - 1) \end{aligned}$$

In this case the following vectors are in L_k :

$$\begin{aligned} -(2a^2 + a + 2)\mathbf{v}_1 + (2a^2 + 2a + 1)\mathbf{v}_2 - a\mathbf{v}_3 - \mathbf{v}_4 &= (4a^3 - 4a^2 + 6a - 1, -1, 0, 0), \\ -(2a^3 - a^2 - 2a - 2)\mathbf{v}_1 + (2a^3 - 4a - 1)\mathbf{v}_2 - (a^2 - a - 1)\mathbf{v}_3 - (a - 1)\mathbf{v}_4 &= (4a^4 - 8a^3 + 8a^2 - 8a, 0, -1, 0), \\ -(2a^3 - a^2 - 2a - 1)\mathbf{v}_1 + (2a^3 - 4a)\mathbf{v}_2 - (a^2 - a - 1)\mathbf{v}_3 - (a - 1)\mathbf{v}_4 &= (4a^4 - 8a^3 + 8a^2 - 6a, 0, 0, -1). \end{aligned}$$

Hence we have $\mathbf{e}_2 = (4a^3 + 4a^2 + 6a - 1)\mathbf{e}_1$, $\mathbf{e}_3 = (4a^4 - 8a^3 + 8a^2 - 8a)\mathbf{e}_1$ and $\mathbf{e}_4 = (4a^4 - 8a^3 + 8a^2 - 6a)\mathbf{e}_1$, in \mathbb{Z}^4/L_k , and so \mathbf{e}_1 generates \mathbb{Z}^4/L_k .

$$\text{Also } \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 - 8a^3 + 12a^2 - 4a & 0 & 0 & 0 \\ 4a^3 - 4a^2 + 6a - 1 & -1 & 0 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 8a & 0 & -1 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 6a & 0 & 0 & -1 \end{pmatrix}$$

$= -(8a^4 - 8a^3 + 12a^2 - 4a) = -(k^4 + 2k^3 + 6k^2 + 6k + 1)/2 = -L(8, k)$, as in the statement of the theorem.

Thus \mathbb{Z}^4/L_k is isomorphic to $\mathbb{Z}_{L(8,k)}$ with generators $1, 4a^3 - 4a^2 + 6a - 1, 4a^4 - 8a^3 + 8a^2 - 8a, 4a^4 - 8a^3 + 8a^2 - 6a$. As $a = (k + 1)/2$ in this case, this gives the second generator set specified in the theorem: $\{1, (k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

It remains to show that $S_k + L_k = \mathbb{Z}^4$. First we consider the case $k \equiv 0 \pmod{2}$.

For $k = 2$, it is straightforward to show directly that \mathbb{Z}_{32} with generators $1, 4, 6, 15$ has

diameter 2. So we assume $k \geq 4$, so that $a \geq 2$. Now let

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_4 = (-a, a, a-1, a+2) \\ \mathbf{v}_6 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = (-a, a, -a-1, -a) \\ \mathbf{v}_7 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = (-a+1, a-1, -a-2, a+1) \\ \mathbf{v}_8 &= \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = (a, a, a, a+1)\end{aligned}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 0 \pmod{2}$. Then the 16 vectors $\pm \mathbf{v}_i$ for $i = 1, \dots, 8$ provide one element of L_k lying strictly within each of the 16 orthants of \mathbb{Z}^4 . Most of the coordinates of these vectors have absolute value at most $a+1$. Only $\pm \mathbf{v}_5$ and $\pm \mathbf{v}_7$ each have one coordinate with absolute value equal to $a+2$.

Now we consider the case $k \equiv 1 \pmod{2}$. For $k = 3$ it may be shown directly that \mathbb{Z}_{104} with generators 1, 16, 20, 27 has diameter 3. So we assume $k \geq 5$, so that $a \geq 3$, and let

$$\begin{aligned}\mathbf{v}_5 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = (-a, -a, -a-1, -a+2) \\ \mathbf{v}_6 &= \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = (a, a, -a+1, -a) \\ \mathbf{v}_7 &= \mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_4 = (-a, a, -a, -a+1) \\ \mathbf{v}_8 &= \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = (-a+1, -a+1, -a, a+1)\end{aligned}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 1 \pmod{2}$, so that the 16 vectors $\pm \mathbf{v}_i$ provide one element of L_k lying strictly within each of the orthants of \mathbb{Z}^4 . In this case all the coordinates of these vectors have absolute value at most $a+1$.

We must show that each $\mathbf{x} \in \mathbb{Z}^4$ is in $S_k + L_k$, which means that for any $\mathbf{x} \in \mathbb{Z}^4$ we need to find a $\mathbf{w} \in L_k$ such that $\mathbf{x} - \mathbf{w} \in S_k$. However $\mathbf{x} - \mathbf{w} \in S_k$ if and only if $\delta(\mathbf{x}, \mathbf{w}) \leq k$, where δ is the l^1 metric on \mathbb{Z}^4 . If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^4$ and each coordinate of \mathbf{y} lies between the corresponding coordinate of \mathbf{x} and \mathbf{z} or is equal to one of them, then $\delta(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{y}, \mathbf{z}) = \delta(\mathbf{x}, \mathbf{z})$. In such a case we say that “ \mathbf{y} lies between \mathbf{x} and \mathbf{z} ”.

For any $\mathbf{x} \in \mathbb{Z}^4$, we reduce \mathbf{x} by adding appropriate elements of L_k until the resulting vector lies within l^1 -distance k of $\mathbf{0}$ or some other element of L_k . The first stage is to reduce \mathbf{x} to a vector whose coordinates all have absolute value at most $a+1$. If \mathbf{x} has a coordinate with absolute value above $a+1$, then let \mathbf{v} be one of the vectors $\pm \mathbf{v}_i$ ($1 \leq i \leq 8$) such that the coordinates of \mathbf{v} have the same sign as the corresponding coordinates of \mathbf{x} . If a coordinate of \mathbf{x} is 0 then either sign is allowed for \mathbf{v} as long as the corresponding coordinate of \mathbf{v} has absolute value $\leq a+1$. So in the case $k \equiv 0 \pmod{2}$ if the \mathbf{e}_3 coordinate of \mathbf{x} is 0 then we avoid \mathbf{v}_7 and take \mathbf{v}_5 instead. Also if the \mathbf{e}_4 coordinate of \mathbf{x} is 0 (or both \mathbf{e}_3 and \mathbf{e}_4 coordinates are 0) then instead of \mathbf{v}_5 we take \mathbf{v}_1 .

Now consider $\mathbf{x}' = \mathbf{x} - \mathbf{v}$. If a coordinate of \mathbf{x} has absolute value s , $1 \leq s \leq a+1$, then the corresponding coordinate of \mathbf{x}' will have absolute value $s' \leq a+1$ because of the sign matching and the fact that the coordinates of \mathbf{v} have absolute value $\leq a+2$. If a coordinate of \mathbf{x} has absolute value $s = 0$, then as indicated above, the corresponding value of \mathbf{x}' will have absolute value $s' \leq a+1$ because \mathbf{v} is chosen such that the corresponding coordinate has absolute value $\leq a+1$. If a coordinate of \mathbf{x} has absolute value $s > a+1$, then the corresponding coordinate of \mathbf{x}' will be strictly smaller in absolute value. Therefore repeating this procedure will result in a vector whose coordinates all have absolute value at most $a+1$.

If the resulting vector \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v} , where $\mathbf{v} = \pm \mathbf{v}_i$ for some i , then we have $\delta(\mathbf{0}, \mathbf{x}') + \delta(\mathbf{x}', \mathbf{v}) = \delta(\mathbf{0}, \mathbf{v})$. For $k \equiv 0 \pmod{2}$ all of the vectors \mathbf{v} satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a + 1$, and for $k \equiv 1 \pmod{2}$ they all satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a - 1$. So in either case we have $\delta(\mathbf{0}, \mathbf{v}) = 2k + 1$. Since $\delta(\mathbf{0}, \mathbf{x}')$ and $\delta(\mathbf{x}', \mathbf{v})$ are both non-negative integers, one of them must be at most k , so that $\mathbf{x}' \in S_k + L_k$. Hence we also have $\mathbf{x} \in S_k + L_k$ as required.

Now we are left with the case where the absolute value of each coordinate of the reduced \mathbf{x} is at most $a + 1$, and \mathbf{x} is in the orthant of \mathbf{v} , where $\mathbf{v} = \pm \mathbf{v}_i$ for some $i \leq 8$ but does not lie between $\mathbf{0}$ and \mathbf{v} . Since L_k is centrosymmetric we only need to consider the eight orthants containing $\mathbf{v}_1, \dots, \mathbf{v}_8$. For both cases $k \equiv 0$ and $k \equiv 1 \pmod{2}$ the exceptions need to be considered for each orthant in turn. We first consider all eight orthants for the case $k \equiv 0 \pmod{2}$ and then the same for $k \equiv 1 \pmod{2}$.

Orthant of \mathbf{v}_1 , $k \equiv 0 \pmod{2}$

Suppose that $k \equiv 0 \pmod{2}$ and \mathbf{x} lies within the orthant of \mathbf{v}_1 , but not between $\mathbf{0}$ and \mathbf{v}_1 . Then as $\mathbf{v}_1 = (-a - 1, a + 1, a, -a + 1)$, the third coordinate of \mathbf{x} is equal to $a + 1$ or the fourth coordinate equals $-a$ or $-a - 1$. We distinguish three cases.

Case 1: $\mathbf{x} = (-r, s, a + 1, -u)$ where $0 \leq r, s \leq a + 1$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, 1, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r \leq 1$ or $s \leq 1$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (2 - r, s - 2, -a - 1, 2a - u)$. If $r \leq 1$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (1 - a - r, a - 1 + s, -1, a + 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r \leq 1$ and $s \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r \geq 2$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 2: $\mathbf{x} = (-r, s, a + 1, -u)$ where $0 \leq r, s \leq a + 1$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, 1, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = 0$ or $s = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (1 - r, s - 1, -a, -u - 1)$. If $r = 0$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $r = 0$ and $s \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r, s \leq a + 1$ and $0 \leq t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, t - a, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $s = 0$ or $t = 0$. If $r = 0$ and $s = 0$, then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_7$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1 - r, s - 1, t - 1, 2a + 1 - u)$. If $r = 0, s \geq 1$ and $t \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_8 . Let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_4 = (-a - r, s - a, a + t, a + 1 - u)$. If $r = 0$ and $s \geq 1$ and $t = 0$, then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 unless $s = a + 1$, in which case if $u = a$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_2 , and if $u = a + 1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 . Let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, a - 1 + s, t - a - 1, a - u)$. If $r \geq 1, s = 0$ and $t \geq 1$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1, s = 0$ and $t = 0$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_3$ if $u = a$, and between $\mathbf{0}$ and \mathbf{v}_6 if $u = a + 1$. If $r \geq 1, s \geq 1$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = a + 1$ or $s = a + 1$. If $r = a + 1, s \geq 1$ and $t = 0$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r \geq 1, s = a + 1$ and $t = 0$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of \mathbf{v}_1 for $k \equiv 0 \pmod{2}$.

Orthant of \mathbf{v}_2 , $k \equiv 0 \pmod{2}$

Now suppose that \mathbf{x} lies in the orthant of \mathbf{v}_2 but not between $\mathbf{0}$ and \mathbf{v}_2 . Then the first coordinate of \mathbf{x} is equal to a or $a+1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $\mathbf{x} = (r, s, t, -a-1)$ where $a \leq r \leq a+1$ and $0 \leq s, t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = 0$ or $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (r-2a+1, s-1, t-2, a+1)$. If $s = 0$ and $t \leq 1$ then let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_5 = (r-a, a, t+a-1, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $s = 0$ and $t \geq 2$ then let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_8 = (r-2a, 0, t-1, -a)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 . If $s \geq 1$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 unless $s = a+1$, in which case let $\mathbf{x}^v = \mathbf{x}'' - \mathbf{v}_7 = (r-a, 1, a+t, 0)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 .

Case 2: $\mathbf{x} = (r, s, t, -u)$ where $a \leq r \leq a+1$, $0 \leq s, t \leq a+1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$. If $t = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $a \leq s \leq a+1$. If $r = a+1$, $a \leq s \leq a+1$, $t = 0$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_3 = (r-a-1, s-a+1, t+a+1, -u-a)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 . If $a \leq r \leq a+1$, $s = a+1$, $t = 0$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -u+a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r = a$, $s = a$, $t = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_8 . Now let $\mathbf{x}''' = \mathbf{x}' + \mathbf{v}_1 = (r-2a, s, t-1, 1-u)$. If $t = 0$ and $1 \leq u \leq a$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_6 unless $s = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_6 = (r-a, s-a, t+a, a+1-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $1 \leq t \leq a+1$ and $u = 0$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_5 unless $s = a+1$ or $t = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_5 = (r-a, s-a, t-a, -a-1-u)$. If $s = a+1$ and $t = a+1$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_8 . If $s = a+1$ and $1 \leq t \leq a$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $0 \leq s \leq a$ and $t = a+1$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 3: $\mathbf{x} = (r, s, t, -a-1)$ where $0 \leq r \leq a-1$ and $0 \leq s, t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = 0$ or $t = 0$. If $s = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $t = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = a+1$, in which case let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_4 = (r-a, 1, a, 0)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 . This completes the cases for the orthant of \mathbf{v}_2 .

Orthant of \mathbf{v}_3 , $k \equiv 0 \pmod{2}$

Now suppose that \mathbf{x} lies in the orthant of \mathbf{v}_3 but not between $\mathbf{0}$ and \mathbf{v}_3 . Then the second coordinate of \mathbf{x} is equal to $-a$ or $-a-1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $\mathbf{x} = (-r, -s, t, -a-1)$ where $0 \leq r, t \leq a+1$ and $a \leq s \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1-r, 2a-1-s, t-2, a+1)$. If $r = 0$ and $t \geq 2$ then \mathbf{x}'' which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $r = 0$ and $t \leq 1$ then let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_5 = (-a, a-s, a-1+t, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $r \geq 1$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_7 = (-1, a-s, a+t, 0)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 2: $\mathbf{x} = (-r, -s, t, -a-1)$ where $0 \leq r, t \leq a+1$ and $0 \leq s \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$ or $t = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (1-r, -1-s, t-1, a)$. If $r = 0$ and $t \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r \geq 1$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $r = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (-1, a-s, a, 0)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 . If $r = 0$ and $t = 0$, then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-a, a-s, a-1, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 3: $\mathbf{x} = (-r, -s, t, -u)$ where $0 \leq r, t \leq a+1$, $a \leq s \leq a+1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$. Let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_8 = (a-r, a-s, a+t, a+1-u)$. If $t = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 unless $r \leq a-1$, in which case let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_2 = (a-1-r, a+1-s, a+1+t, -a-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 . If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = a+1$, in which case \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 . Let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_4 = (a-r, a-s, t-a, -a-1-u)$. If $t \geq 1, u = 0$ and $r \leq a$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a+1$. If $t = a+1, u = 0$ and $r \leq a$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ in which case \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_8 . If $t \geq 1, u = 0$ and $r = a+1$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

This completes the cases for the orthant of \mathbf{v}_3 .

Orthant of \mathbf{v}_4 , $k \equiv 0 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_4 but not between $\mathbf{0}$ and \mathbf{v}_4 . Then the first coordinate of \mathbf{x} is equal to $-a-1$ or the second coordinate is equal to $-a-1$, or the third equals $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, -a-1, a+1, u)$ where $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -1, 1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 if $u = a+1$ and between $\mathbf{0}$ and \mathbf{v}_3 if $u \leq a$.

Case 2: $\mathbf{x} = (-a-1, -a-1, t, u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-a-1, -s, a+1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -a-s, 1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u \geq a$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -s-1, -a+1, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 4: $\mathbf{x} = (-r, -a-1, a+1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, -1, 1, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1, a-2, -a-1, u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a, -1, 0, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 5: $\mathbf{x} = (-r, -s, a+1, u)$ where $0 \leq r, s \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, a-s, 1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $r = 0$ or $u = 0$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -s-1, -a, u-1)$. If $r \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a+1-r, -1, 0, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = 0$ or $u = a+1$. If $r = 0$ and $u = 0$, then let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a, a-1-s, -1, a+1)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 unless $s = a$, in which case let $\mathbf{x}^v = \mathbf{x}'''' + \mathbf{v}_2 = (0, a, a, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $r = 0$ and $u = a+1$, then let $\mathbf{x}^{vi} = \mathbf{x}'' + \mathbf{v}_1 = (-a, a-s, 0, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 6: $\mathbf{x} = (-r, -a-1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, -1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-r, a-1, -1, u+1)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = a$ or $u = a+1$. If $t = 0$ and $r = a$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (-1, 0, a+1, u-a)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $u = a+1$. If $t = 0$ and $u = a+1$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (a-1-r, 0, a+1, 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = a$. If $t = 0, r = a$ and $u = a+1$, then $\mathbf{x}''' = (-1, 0, a+1, 1)$. Let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_4 = (a-1, a, 1, -a)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 .

Case 7: $\mathbf{x} = (-a-1, -s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, a-s, t-a, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a-1, -s, t+1, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_7 = (0, -1, t-a-1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

This completes the cases for the orthant of \mathbf{v}_4 .

Orthant of \mathbf{v}_5 , $k \equiv 0 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_5 but not between $\mathbf{0}$ and \mathbf{v}_5 . Then the first coordinate of \mathbf{x} is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals a or $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, a+1, t, u)$ where $a \leq t \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u \leq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, t-2a+1, u-3)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \leq t \leq a-1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a-1, -a+1, t+2, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 3: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \leq s \leq a$, $a \leq t \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, s-a, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = 0$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, s-1, t-2a, u-2)$. If $s = 0$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, a-1, -1, a+u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 . If $s = 0$ and $u \geq 2$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, a, t-a, u-a-1)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 . If $s \geq 1$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \leq r \leq a$, $a \leq t \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $r = 0$ or $u \geq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -a, t-2a, u-2)$. If $r = 0$ and $u \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $r = 0$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a, 0, -1, a+u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r \geq 1$ and $u \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, t, u)$ where $0 \leq r, s \leq a$, $a \leq t \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, s-a, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $s = 0$ or $u = 0$. If $r = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_8 unless $t = a+1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (-a, s-a, 1, u-a-1)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $u = 0$. If $r = 0, t = a+1$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_2 . If $r \geq 1$ and $s = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_4 unless $t = a+1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, a, 1, u-a-1)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $u = 0$. If $r \geq 1, s = 0$ and $u = 0$, then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -1, -a, -1)$ which lies between $\mathbf{0}$

and $-\mathbf{v}_8$. If $r \geq 1, s \geq 1$ and $u = 0$, then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 unless $t = a + 1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, s - a - 1, 1, a - 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 6: $\mathbf{x} = (-r, a + 1, t, u)$ where $0 \leq r \leq a, 0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, 1, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $u = 0$ in which case \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 7: $\mathbf{x} = (-a - 1, s, t, u)$ where $0 \leq s \leq a, 0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, s - a, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $u = 0$ in which case \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 .

This completes the cases for the orthant of \mathbf{v}_5 .

Orthant of $\mathbf{v}_6, k \equiv 0 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_6 but not between $\mathbf{0}$ and \mathbf{v}_6 . Then the first coordinate of \mathbf{x} is equal to $-a - 1$ or the second coordinate is equal to $a + 1$, or the fourth equals $-a - 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a - 1, a + 1, -t, -a - 1)$ where $0 \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, 1, a + 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, 1, a - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 2: $\mathbf{x} = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, 1, a + 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (a - 1, 1 - a, 2 - t, -u - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = a$. If $t = 1$ and $u = a$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_1 . If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, 1, a + 1)$ and $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_6 = (0, 0, -a, 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 3: $\mathbf{x} = (-a - 1, s, -t, -a - 1)$ where $0 \leq s \leq a$ and $0 \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, s - a, a + 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = a$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, a - 1, -t, a - 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 4: $\mathbf{x} = (-r, a + 1, -t, -a - 1)$ where $0 \leq r \leq a$ and $0 \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a - r, 1, a + 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1, -a, -t, -a - 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = a + 1$ in which case $\mathbf{x}' = (a, 1, 0, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 5: $\mathbf{x} = (-r, s, -t, -a - 1)$ where $0 \leq r, s \leq a$ and $0 \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a - r, s - a, a + 1 - t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1 - r, s - 1, -t - 1, a)$. If $r = 0$ and $s \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a + 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a, s - a, -1, 0)$ which lies between $\mathbf{0}$ and \mathbf{v}_6 . If $r \geq 1$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a + 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a - r, a, -1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r = 0$ and $s = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $t = a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x} + \mathbf{v}_8 = (a, a, -1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 6: $\mathbf{x} = (-r, a + 1, -t, -u)$ where $0 \leq r, u \leq a$ and $0 \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a - r, 1, a + 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $t = 0$ in which case let $\mathbf{x} = (-r, a + 1, 0, -u)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u = a$. If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_1 = (a + 1 - r, 0, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$ in

which case $\mathbf{x} = (0, a+1, 0, -a)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 7: $\mathbf{x} = (-a-1, s, -t, -u)$ where $0 \leq s, u \leq a$ and $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, s-a, a+1-t, a-u)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 unless $t = 0$ in which case $\mathbf{x} = (-a-1, s, 0, -u)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u = a$. If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_1 = (0, s-a-1, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = 0$ in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a, -1, 0, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

This completes the cases for the orthant of \mathbf{v}_6 .

Orthant of \mathbf{v}_7 , $k \equiv 0 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_7 but not between $\mathbf{0}$ and \mathbf{v}_7 . Then the first coordinate of \mathbf{x} is equal to $-a$ or $-a-1$ or the second equals a or $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r, s \leq a+1$, $2 \leq t \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_6 = (a-r, s-a, a+1-t, a+u)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 2: $\mathbf{x} = (-a, a, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a+1$. If $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_6 . If $u \geq 1$ then let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 - \mathbf{v}_3 = (a, a, 1-t, u-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 .

Case 3: $\mathbf{x} = (-a-1, a, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (0, -1, -a-t, a-1+u)$. If $u \leq 1$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $u \geq 2$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a-1, a, 1-t, u-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 .

Case 4: $\mathbf{x} = (-a, a+1, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (1, 0, -a-t, a-1+u)$. If $u \leq 1$ then \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $u \geq 2$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-a, -a+1, 1-t, u-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 5: $\mathbf{x} = (-a-1, a+1, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (0, 0, -a-t, a-1+u)$. If $u \leq 1$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_7 . If $u \geq 2$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (a-1, -a+1, 2-t, u-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 6: $\mathbf{x} = (-r, s, -t, u)$ where $0 \leq r \leq a-1$, $a \leq s \leq a+1$ and $0 \leq t, u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r, s-2a, 1-t, u-1)$. If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 . If $t = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_3 = (a+1-r, -1, -a, a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $t \geq 1$ and $u = 0$ then $\mathbf{x}'' = (-r, s-2a, 1-t, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 7: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r \leq a+1$, $0 \leq s \leq a-1$ and $0 \leq t, u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (2a-r, s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 . If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_8 . If $t \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of \mathbf{v}_7 .

Orthant of \mathbf{v}_8 , $k \equiv 0 \pmod{2}$

Finally suppose \mathbf{x} lies in the orthant of \mathbf{v}_8 but not between $\mathbf{0}$ and \mathbf{v}_8 . Then at least one of the first three coordinate of \mathbf{x} is equal to $a + 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (a + 1, a + 1, a + 1, u)$ where $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, 1, 1, u - a - 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-a + 2, -a, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 2: $\mathbf{x} = (a + 1, a + 1, t, u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, 1, t - a, u - a - 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 3: $\mathbf{x} = (a + 1, s, a + 1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, s - a, 1, u - a - 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (-a + 2, -1, -a - 1, u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a + 1$. If $s = 0$ and $u = a + 1$ then $\mathbf{x}' = (1, -a, 1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 4: $\mathbf{x} = (r, a + 1, a + 1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r - a, 1, 1, u - a - 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (r + 1, -a, -a - 1, u - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = a$. If $r = a$ and $u \leq 1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1, 0, 0, u + a)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 .

Case 5: $\mathbf{x} = (a + 1, s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, s - a, t - a, u - a - 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-a + 1, s, -1, u + 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 unless $s = a$ or $u = a + 1$. If $s = a$ and $t = 0$ then $\mathbf{x}' = (1, 0, -a, u - a - 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $t = 0$ and $u = a + 1$ then $\mathbf{x}' = (1, s - a, -a, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 6: $\mathbf{x} = (r, a + 1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r - a, 1, t - a, u - a - 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $u = 0$, in which case \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_2 unless $r = a$. If $r = a$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_2 = (1, 0, t - a - 1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 7: $\mathbf{x} = (r, s, a + 1, u)$ where $0 \leq r, s \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r - a, s - a, 1, u - a - 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (r + 1, s - 1, -a, u - 1)$. If $s = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $s = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a + 1$, in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_4 . If $s \geq 1$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_2 unless $r = a$, in which case \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

This completes the cases for the orthant of \mathbf{v}_8 .

This also completes the proof of the theorem for any $k \equiv 0 \pmod{2}$.

Now we consider the eight orthants $\mathbf{v}_1, \dots, \mathbf{v}_8$ in turn for the case $k \equiv 1 \pmod{2}$.

Orthant of \mathbf{v}_1 , $k \equiv 1 \pmod{2}$

First suppose that \mathbf{x} lies within the orthant of \mathbf{v}_1 , but not between $\mathbf{0}$ and \mathbf{v}_1 . Then the first coordinate of \mathbf{x} is equal to $-a$ or $-a - 1$, or the third coordinate equals $-a$ or $-a - 1$, or the fourth equals $a + 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, -t, a + 1)$ where $a \leq r, t \leq a + 1$ and $0 \leq s \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 =$

$(a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $s \leq 1$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (2a - 2 - r, s - 2, 2a - 1 - t, -a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 2: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r, t \leq a + 1$ and $0 \leq s \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $s = 0$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (2a - 1 - r, s - 1, 2a - t, u - 2)$. If $s = 0$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$, unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (a - r, a, 1, u + a - 2)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $s = 0$ and $u \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $s \geq 1$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_8 = (a - r, 1, a - t, a + u - 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 3: $\mathbf{x} = (-r, s, -t, a + 1)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a + 1$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (2a - 1 - r, -1, -t, -a + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, s, -t, a + 1)$ where $0 \leq r \leq a - 1$, $0 \leq s \leq a + 1$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-2 - r, s - 2, 2a - 2 - t, -a + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, -t, a + 1)$ where $0 \leq r, t \leq a - 1$ and $0 \leq s \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (-r - 1, -1, -t - 1, -a + 2)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 6: $\mathbf{x} = (-r, s, -t, u)$ where $0 \leq r \leq a - 1$, $0 \leq s \leq a + 1$, $a \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-r - 1, s - 1, 2a - 1 - t, u - 1)$. If $s = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a - r, a, a + 1 - t, a - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $s = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $s \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = a + 1$, in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 7: $\mathbf{x} = (-r, s, -t, u)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a + 1$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a - 1 - r, s - a - 1, a - 1 - t, u - a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (2a - r, s, -t + 1, u - 1)$. If $t = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a + 1 - r, s - a + 1, -a + 1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $a \leq s \leq a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_7 = (a - r, s - a, a, a - 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a + 1$ or $u = a$ in which case let $\mathbf{x}^v = \mathbf{x}'' + \mathbf{v}_5 = (a - r, s - a, -a, -a + u + 1)$. If $s = a + 1$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_3 . If $1 \leq s \leq a$ and $u = a$ then \mathbf{x}^v lies between $\mathbf{0}$ and \mathbf{v}_8 . If $s = 0$ and $u = a$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $t \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_6 unless $s = a + 1$, in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_3 .

This completes the cases for the orthant of \mathbf{v}_1 .

Orthant of \mathbf{v}_2 , $k \equiv 1 \pmod{2}$

Now suppose that \mathbf{x} lies in the orthant of \mathbf{v}_2 but not between $\mathbf{0}$ and \mathbf{v}_2 . Then the third coordinate of \mathbf{x} is equal to $-a + 1$, $-a$ or $-a - 1$, or the fourth coordinate equals a or $a + 1$.

We distinguish three cases.

Case 1: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $a-1 \leq t \leq a+1$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r \leq 1$ or $s \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (r-2, s-2, 2a-2-t, u-2a)$. If $r \leq 1$ and $s \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_3 . If $r \geq 2$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $r \leq 1$ and $s \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a-1$ or $u = a$. Let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (r+a-1, s+a-1, a-t, u-a-1)$. If $r \leq 1$ and $s \leq 1$ and $u = a$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_6 unless $t = a-1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_6 = (r-1, s-1, 2a-1-t, u-1)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 if $s = 1$, and between $\mathbf{0}$ and $-\mathbf{v}_7$ if $r = 1$. If $r = 0$ and $s = 0$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 . If $r \leq 1$ and $s \leq 1$ and $t = a-1$ and $u = a+1$ then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $0 \leq t \leq a-2$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = 0$ or $s = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (r-1, s-1, -t-1, u-2a+1)$. If $r = 0$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_5 unless $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_5 = (r+a-1, s+a-1, a-t, u-a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0$ and $s \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r \geq 1$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 3: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $a-1 \leq t \leq a+1$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $r = 0$ or $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (r-1, s-1, 2a-1-t, u-1)$. If $r = 0$ and $s = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $a-1 \leq t \leq a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a+r, a+s, a+1-t, a-2+u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $r = 0$ and $s = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t = a-1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_6 = (r+a-1, s+a-1, a-t, u-a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0$ and $s \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = a+1$ or $t = a-1$, in which case let $\mathbf{x}^v = \mathbf{x}'' - \mathbf{v}_3 = (r+a, s-a, a-t, u+a-1)$. If $s = a+1$ then \mathbf{x}^v lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = a-1$. If $t = a-1$ then \mathbf{x}^v lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = a+1$, in which case let $\mathbf{x}^{vi} = \mathbf{x}^v + \mathbf{v}_5 = (r, s-2a, -t-1, u+1)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $r = 0$ and $s \geq 1$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 . If $r \geq 1$ and $s = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = a+1$ or $t = a-1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (r-a, s+a, a-t, a+u-1)$. If $r = a+1$ and $t \geq a$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_2 . If $r = a+1$ and $t = a-1$ then let $\mathbf{x}'''' = \mathbf{x}''' + \mathbf{v}_5 = (r-2a, s, -t-1, u+1)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $1 \leq r \leq a$ and $t = a-1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_4 . If $r \geq 1$ and $s = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $r \geq 1$ and $s \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = a+1$ or $s = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (r-a, s-a, a-1-t, u+a)$. If $r = a+1$ and $s = a+1$ then let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_2 = (r-2a-1, s-2a-1, 2a-3-t, u+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = a+1$ and $1 \leq s \leq a$ then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $1 \leq r \leq a$ and $s = a+1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_1 .

This completes the cases for the orthant of \mathbf{v}_2 .

Orthant of \mathbf{v}_3 , $k \equiv 1 \pmod{2}$

Now suppose that \mathbf{x} lies in the orthant of \mathbf{v}_3 but not between $\mathbf{0}$ and \mathbf{v}_3 . Then the second coordinate of \mathbf{x} is equal to a or $a+1$, or the third coordinate equals a or $a+1$, or the

fourth equals $-a - 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $a \leq s, t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (2 - r, s - 2a + 2, t - 2a + 1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 2: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$ and $a \leq s, t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1 - r, s - 2a + 1, t - 2a, 2 - u)$. If $r = 0$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a - r, s - a, t - a - 1, 2 - a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r = 0$ and $u \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r = a + 1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_8 = (a - r, s - a, t - a, 1 - a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 3: $\mathbf{x} = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$, $a \leq s \leq a + 1$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (1 - r, s - 2a + 1, t, a - 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $0 \leq s \leq a - 1$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (2 - r, s + 2, t - 2a + 2, a - 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 .

Case 5: $\mathbf{x} = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $0 \leq s, t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (1 - r, s + 1, t + 1, a - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$, $0 \leq s \leq a - 1$, $a \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1 - r, s + 1, t - 2a + 1, 1 - u)$. If $r = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a - r, s - a, t - a - 1, 2 - a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 . If $r = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_6 . If $r \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_1 unless $r = a + 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_1 = (a - r, s - a, t - a, 1 - a - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 7: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$, $a \leq s \leq a + 1$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a + 1 - r, s - a + 1, t - a + 1, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r, s - 2a, t - 1, 1 - u)$. If $t = 0$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r \leq 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a - 1 - r, s - a - 1, t + a - 1, -a - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $t = 0$ and $u \geq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_5 unless $r = 0$ or $u = a$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_5 = (a - r, s - a, a + t, a - u - 1)$. If $r = 0$, $t = 0$ and $u = a$ then let $\mathbf{x}^v = \mathbf{x}'''' + \mathbf{v}_8 = (1 - r, s - 2a + 1, t, 2a - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r = 0$, $t = 0$ and $1 \leq u \leq a - 1$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \leq r \leq a$, $t = 0$ and $u = a$ then \mathbf{x}'''' lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = a + 1$, $t = 0$ and $u = a$ then \mathbf{x}^v lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $t \geq 1$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

This completes the cases for the orthant of \mathbf{v}_3 .

Orthant of \mathbf{v}_4 , $k \equiv 1 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_4 but not between $\mathbf{0}$ and \mathbf{v}_4 . Then the first coordinate of \mathbf{x} is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals $a+1$ or the fourth equals a or $a+1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (-a-1, a+1, a+1, u)$ where $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 2: $\mathbf{x} = (-a-1, a+1, a+1, u)$ where $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 3: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 4: $\mathbf{x} = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, s, a+1, u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 7: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \leq r, t \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r-1, -a, t-2, u)$ and $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-r-2, -1, a+t-2, u-a-1)$. \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $a-1 \leq r \leq a$, in which case let $\mathbf{x}'''' = \mathbf{x}''' + \mathbf{v}_2 = (2a-r-1, a, t, u-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a+1$. If $a-1 \leq r \leq a$, $t \leq 1$ and $u = a+1$, then let $\mathbf{x}^v = \mathbf{x}'''' + \mathbf{v}_5 = (a-r-1, 0, t-a-1, u-a)$ and $\mathbf{x}^{vi} = \mathbf{x}^v - \mathbf{v}_8 = (2a-r-2, a-1, t-1, u-2a-1)$ which lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 8: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \leq s, t \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, t-a, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (a-2, s-1, t, u-2a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = a$. If $s = 0$ and $t = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, s+a, t-a+1, u-a)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 9: $\mathbf{x} = (-r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1-r, 2-a, 1-a, u+2)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 .

Case 10: $\mathbf{x} = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 11: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 12: $\mathbf{x} = (-r, s, t, u)$ where $0 \leq r, s, t \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, s-a, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$ or $t = 0$. If $s = 0$ and $t = 0$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-1-r, s-1, t-1, u-2a+1)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $r = a$ or $u = a$ in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_5 = (a-1-r, s+a-1, t+a, u-a-1)$. If $r = a$ and $u = a$, then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = a$ and $u = a+1$, then \mathbf{x}'''

lies between $\mathbf{0}$ and \mathbf{v}_4 . If $r \leq a - 1$ and $u = a$, then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $s = 0$ and $1 \leq t \leq a$ then let \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a - r, a + s, t - a + 1, u - a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \leq s \leq a$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (a - 1 - r, s - a - 1, t + a - 1, u - a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 13: $\mathbf{x} = (-r, s, a + 1, u)$ where $0 \leq r, s \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a - r, s - a, 1, u - a + 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1 - r, s + 1, -a + 2, u + 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $u = a - 1$. If $r = 0$ and $u = a - 1$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a - r, s - a, 0, u - a + 2)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 14: $\mathbf{x} = (-r, a + 1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a - r, 1, t - a, u - a + 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (-r, 1 - a, t - 1, u + 1)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r = a$. If $r = a$ and $t = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a - 1 - r, 0, t + a - 1, u - a)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 15: $\mathbf{x} = (-a - 1, s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s - a, t - a, u - a + 1)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $t = 0$ or $u = 0$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (a - 1, s, t + 1, u - 1)$. If $t = 0$ and $u = 0$, then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (0, s - a + 1, t - a + 1, u + a)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 . If $t = 0$ and $1 \leq u \leq a - 1$, then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \leq t \leq a - 1$ and $u = 0$, then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_1 . If $t = a$ and $u = 0$ then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $s = a$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_4 = (a, s - 2a + 1, t - 2a + 1, u + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

This completes the cases for the orthant of \mathbf{v}_4 .

Orthant of \mathbf{v}_5 , $k \equiv 1 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_5 but not between $\mathbf{0}$ and \mathbf{v}_5 . Then the first coordinate of \mathbf{x} is equal to $-a - 1$ or the second coordinate is equal to $-a - 1$, or the fourth equals $-a + 1$, $-a$ or $-a - 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a - 1, -a - 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, -1, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \leq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a, a, 3 - t, 2a - 3 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a - 1, -a - 1, -t, -u)$ where $0 \leq t \leq a + 1$ and $0 \leq u \leq a - 2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, -1, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (a - 1, a - 1, 2 - t, -2 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-a - 1, -s, -t, -u)$ where $0 \leq s \leq a$, $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, a - s, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $s = 0$ or $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, 1 - s, 2 - t, 2a - 2 - u)$. If $s = 0$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a - 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, 1 - a - s, 1 - a - t, a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $s = 0$ and $t \geq 2$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = a + 1$, in which case let

$\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-1, -a - s, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $1 \leq s \leq a$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-r, -a - 1, -t, -u)$ where $0 \leq r \leq a$, $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, -1, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$ or $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1 - r, a, 2 - t, 2a - 2 - u)$. If $r = 0$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a - 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1 - a - r, 0, 1 - a - t, a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $r = 0$ and $2 \leq t \leq a + 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = a + 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a - r, -1, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 . If $r \geq 1$ and $t \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 5: $\mathbf{x} = (-r, -s, -t, -u)$ where $0 \leq r, s \leq a$, $0 \leq t \leq a + 1$ and $a - 1 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, a - s, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$ or $s = 0$ or $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1 - r, 1 - s, 1 - t, 2a - 1 - u)$. If $r = 0$, $s = 0$ and $t = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0$, $s = 0$ and $1 \leq t \leq a + 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $a \leq t \leq a + 1$ or $u = a - 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a - r, -a - s, a - 1 - t, a - u)$. If $a \leq t \leq a + 1$ and $a \leq u \leq a + 1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_5 . If $1 \leq t \leq a - 1$ and $u = a - 1$, then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $a \leq t \leq a + 1$ and $u = a - 1$, then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_7 = (a - r, -a - s, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 6: $\mathbf{x} = (-r, -a - 1, -t, -u)$ where $0 \leq r \leq a$, $0 \leq t \leq a + 1$ and $0 \leq u \leq a - 2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, -1, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_7 = (-r, a - 1, 1 - t, -1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 7: $\mathbf{x} = (-a - 1, -s, -t, -u)$ where $0 \leq s \leq a$, $0 \leq t \leq a + 1$ and $0 \leq u \leq a - 2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, a - s, a + 1 - t, a - 2 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_4 = (a - 1, -s, 1 - t, -1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

This completes the cases for the orthant of \mathbf{v}_5 .

Orthant of \mathbf{v}_6 , $k \equiv 1 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_6 but not between $\mathbf{0}$ and \mathbf{v}_6 . Then the first coordinate of \mathbf{x} is equal to $a + 1$ or the second coordinate is equal to $a + 1$, or the third equals $-a$ or $-a - 1$ or the fourth equals $-a - 1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (a + 1, a + 1, -t, -a - 1)$ where $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 2: $\mathbf{x} = (a + 1, a + 1, -t, -u)$ where $a \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-a, -a, 2a - 3 - t, 1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 3: $\mathbf{x} = (a + 1, a + 1, -t, -a - 1)$ where $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 4: $\mathbf{x} = (a + 1, s, -t, -a - 1)$ where $0 \leq s \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s - a, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 5: $\mathbf{x} = (r, a + 1, -t, -a - 1)$ where $0 \leq r \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 =$

$(r - a, 1, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 6: $\mathbf{x} = (r, s, -t, -a - 1)$ where $0 \leq r, s \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, s - a, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 7: $\mathbf{x} = (r, a + 1, -t, -a - 1)$ where $0 \leq r \leq a$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, 1, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 8: $\mathbf{x} = (a + 1, s, -t, -a - 1)$ where $0 \leq s \leq a$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s - a, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 9: $\mathbf{x} = (r, a + 1, -t, -u)$ where $0 \leq r, u \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, 1, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (r - 1, -a, 2a - 2 - t, -u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a$ in which case \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 10: $\mathbf{x} = (a + 1, s, -t, -u)$ where $0 \leq s, u \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s - a, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-a, s - 1, 2a - 2 - t, -u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a$. If $s = 0$ and $u = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (1, s + a, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_2 .

Case 11: $\mathbf{x} = (a + 1, a + 1, -t, -u)$ where $0 \leq t \leq a - 1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $a - 1 \leq u \leq a$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1 - a, 1 - a, -2 - t, 2 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 12: $\mathbf{x} = (r, s, -t, -a - 1)$ where $0 \leq r, s \leq a$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, s - a, a - 1 - t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (r + 1, s + 1, 1 - t, a - 2)$ and $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (r - a + 1, s - a + 1, -a - t, 0)$. Then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_8 .

Case 13: $\mathbf{x} = (r, s, -t, -u)$ where $0 \leq r, s, u \leq a$ and $a \leq t \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, s - a, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r = 0$ or $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (r - 1, s - 1, 2a - 1 - t, -1 - u)$. If $r = 0$ and $s = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a$ or $a - 1 \leq u \leq a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a, a, a + 1 - t, a - 2 - u)$. If $t = a$ and $0 \leq u \leq a - 2$ then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $a \leq t \leq a + 1$ and $a - 1 \leq u \leq a$ then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_4 = (a, -a, t - a, u - a + 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $r = 0$ and $1 \leq s \leq a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_7 = (r + a, s - a, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $u = a$, in which case \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $1 \leq r \leq a$ and $s = 0$ then let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_4 = (r - a, s + a, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $u = a$, in which case \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 14: $\mathbf{x} = (r, a + 1, -t, -u)$ where $0 \leq r, u \leq a$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r - a, 1, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_4 = (r, 1 - a, -1 - t, 1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a - 1$. If $t = a - 1$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (r - a - 1, 0, a - 2 - t, 1 - a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 unless $r = 0$, in which case \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 15: $\mathbf{x} = (a + 1, s, -t, -u)$ where $0 \leq s, u \leq a$ and $0 \leq t \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s - a, a - 1 - t, a - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1 - a, s, -1 - t, 1 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 unless $t = a - 1$. If $t = a - 1$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_1 = (0, s - a - 1, a - 2 - t, 1 - a - u)$ which lies

between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = 0$, in which case \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

This completes the cases for the orthant of \mathbf{v}_6 .

Orthant of \mathbf{v}_7 , $k \equiv 1 \pmod{2}$

Now suppose \mathbf{x} lies in the orthant of \mathbf{v}_7 but not between $\mathbf{0}$ and \mathbf{v}_7 . Then the first coordinate of \mathbf{x} is equal to $-a - 1$ or the second is equal to $a + 1$ or the third equals $-a - 1$, or the fourth equals $-a$ or $-a - 1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (-a - 1, a + 1, -a - 1, -u)$ where $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 2: $\mathbf{x} = (-a - 1, a + 1, -a - 1, -u)$ where $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_1 .

Case 3: $\mathbf{x} = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, a - t, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, 2 - a, 1 - t, 2a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-a - 1, s, -a - 1, -u)$ where $0 \leq s \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s - a, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 5: $\mathbf{x} = (-r, a + 1, -a - 1, -u)$ where $0 \leq r \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a - r, 1, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 6: $\mathbf{x} = (-r, s, -a - 1, -u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a - r, s - a, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 7: $\mathbf{x} = (-r, a + 1, -t, -u)$ where $0 \leq r, t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a - r, 1, a - t, a - 1 - u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1 - r, 2 - a, -t, 2a - u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a$. If $r = 0$ and $t = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a - r, 1, a - 1 - t, a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 8: $\mathbf{x} = (-a - 1, s, -t, -u)$ where $0 \leq s, t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s - a, a - t, a - 1 - u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a, s + 1, 2 - t, 2a - 2 - u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a$. If $s = a$ and $t \leq 1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, s - a + 1, -a + 1 - t, a - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 9: $\mathbf{x} = (-r, a + 1, -a - 1, -u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a - r, 1, -1, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_2 .

Case 10: $\mathbf{x} = (-a - 1, s, -a - 1, -u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s - a, -1, a - 1 - u)$ lies between $\mathbf{0}$ and \mathbf{v}_8 unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (a - 2, s - 1, a - 1, -2 - u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a - 1$. If $s = 0$ and $u = a - 1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, s + a, 0, a - 2 - u)$ which lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 11: $\mathbf{x} = (-a - 1, a + 1, -t, -u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a - 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, a - t, a - 1 - u)$, which lies between $\mathbf{0}$ and \mathbf{v}_4 .

Case 12: $\mathbf{x} = (-r, s, -t, -u)$ where $0 \leq r, s, t \leq a$ and $a \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a - r, s - a, a - t, a - 1 - u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$ or $t = 0$, in which case let

$\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1-r, s+1, 1-t, 2a-1-u)$. If $r = 0$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a$ or $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a-r, s-a+1, -a-t, a+1-u)$. If $u = a$ then \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_6 . If $s = a$ and $u = a+1$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_7 . If $r = 0$ and $1 \leq t \leq a$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = a$, in which case \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $1 \leq r \leq a$ and $t = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 unless $s = a$, in which case \mathbf{x}' lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 13: $\mathbf{x} = (-r, s, -a-1, -u)$ where $0 \leq r, s \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, s-a, -1, a-1-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-1-r, s-1, a-2, -1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a-1$. If $s = 0$ and $u = a-1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a-r, a+s, 0, a-2-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 14: $\mathbf{x} = (-r, a+1, -t, -u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, 1, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-r, -a+1, -t-1, 1-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 unless $r = a$ or $t = a$. If $r = a$ and $u = 0$ then \mathbf{x}' lies between $\mathbf{0}$ and \mathbf{v}_4 . If $0 \leq r \leq a-1$, $t = a$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-1-r, 0, a-1-t, -a-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 15: $\mathbf{x} = (-a-1, s, -t, -u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s-a, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (a-1, s, 1-t, -1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$. If $s = a$ and $t = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (0, s-a+1, 1-a-t, a-u)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 .

This completes the cases for the orthant of \mathbf{v}_7 .

Orthant of \mathbf{v}_8 , $k \equiv 1 \pmod{2}$

Finally suppose \mathbf{x} lies in the orthant of \mathbf{v}_8 but not between $\mathbf{0}$ and \mathbf{v}_8 . Then the first coordinate of \mathbf{x} is equal to $-a$ or $-a-1$, or the second is equal to $-a$ or $-a-1$, or the third is equal to $-a-1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, -s, -a-1, u)$ where $a \leq r, s \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_5 unless $u \leq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (2a-1-r, 2a-1-s, a, u-3)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 2: $\mathbf{x} = (-r, -s, -t, u)$ where $a \leq r, s \leq a+1$, $0 \leq t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \leq 1$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (2a-r, 2a-s, 2-t, u-2)$. If $t \leq 1$ and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = a$ or $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a+1-r, a+1-s, 2-a-t, u+a-1)$. If $t \leq 1$, $u \leq 1$, $r = a$ and $s = a$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_2 unless $t = 1$ or $u = 1$. If $r = a$, $s = a$, $t = 1$ and $u \leq 1$ then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_5 = (a-r, a-s, a+1-t, a-2+u)$ which lies between $\mathbf{0}$ and \mathbf{v}_4 . If $r = a$, $s = a$, $t = 0$ and $u = 1$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $t \leq 1$, $u \leq 1$, $r = a$ and $s = a+1$ then \mathbf{x}''' lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $t \leq 1$, $u \leq 1$, $r = a+1$ and $s = a$ then \mathbf{x}''' lies between $\mathbf{0}$ and \mathbf{v}_1 . If $t \leq 1$ and $2 \leq u \leq a+1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_5 = (a-r, a-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and \mathbf{v}_8 . If $2 \leq t \leq a+1$, and $u \leq 1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_6 .

Case 3: $\mathbf{x} = (-r, -s, -a-1, u)$ where $a \leq r \leq a+1$, $0 \leq s \leq a-1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_7 unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (2a-1-r, -1-s, a-1, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 4: $\mathbf{x} = (-r, -s, -a-1, u)$ where $0 \leq r \leq a-1$, $a \leq s \leq a+1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-1-r, 2a-1-s, a-1, u-2)$ which lies between $\mathbf{0}$ and \mathbf{v}_3 .

Case 5: $\mathbf{x} = (-r, -s, -a-1, u)$ where $0 \leq r, s \leq a-1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_6 unless $u = 0$, in which case \mathbf{x} lies between $\mathbf{0}$ and \mathbf{v}_5 .

Case 6: $\mathbf{x} = (-r, -s, -t, u)$ where $0 \leq r \leq a-1$, $a \leq s \leq a+1$, $0 \leq t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (-r, 2a-2-s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $t = 0$ and $1 \leq u \leq a+1$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_4 unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_4 = (a-r, a-2-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. . If $1 \leq t \leq a$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and \mathbf{v}_7 .

Case 7: $\mathbf{x} = (-r, -s, -t, u)$ where $a \leq r \leq a+1$, $0 \leq s \leq a-1$, $0 \leq t \leq a$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and \mathbf{v}_3 unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (2a-r, -s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then \mathbf{x} lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $t = 0$ and $1 \leq u \leq a+1$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_7 = (a-r, a-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and \mathbf{v}_1 . . If $1 \leq t \leq a$ and $u = 0$ then \mathbf{x}'' lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of \mathbf{v}_8 .

This also completes the proof of the theorem for any $k \equiv 1 \pmod{2}$, and therefore for all $k \geq 2$. \square

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